MATH5636 Stochastic Calculus for Finance II Spring 2026

Chapter 1: Fundamental Theorems of Asset Pricing

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Preview

This chapter continues the discussion of risk-neutral pricing from MATH5635. We begin by constructing hedging portfolios for contingent claims using the martingale representation theorem in a one-dimensional market. We then extend the framework to multi-dimensional market models with multiple risky assets and driving Brownian motions. In this context, we explore how the number of traded assets relative to the number of driving Brownian motions affects the construction of risk-neutral measures (RNMs), market completeness, and arbitrage opportunities, all within the framework of the fundamental theorems of asset pricing.

Key topics in this chapter:

1. Martingale representation theorem;
2. Multi-dimensional market model;
3. The first and second fundamental theorems of asset pricing.

1 Martingale Representation Theorem

In this section, we let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space, and $\{B_t\}_{t\in[0,T]}$ be a standard Brownian motion. We consider a single risky asset whose price $\{S_t\}_{t\in[0,T]}$ is governed by

$$dS_t = S_t\mu_t\,dt + S_t\sigma_t\,dB_t, \tag{1}$$

where μ, σ are $\{\mathcal{F}_t\}_{t\in[0,T]}$-adapted, and $\sigma_t > 0$ a.s. We also let $\{r_t\}_{t\in[0,T]}$ be a $\{\mathcal{F}_t\}_{t\in[0,T]}$-adapted process representing the risk-free interest rate, and $\{D_t\}_{t\in[0,T]}$ be the process of discount factor, given by

$$D_t = e^{-\int_0^t r_s\,ds}.$$

1.1 Review of Risk-Neutral Pricing

Recall that in risk-neutral pricing, we constructed a risk-neutral measure (RNM) such that one can price the contingent claim with payoff V_T at time T, where V_T is $\mathcal{F}_T$-measurable, using the formula

$$V_0 = \widetilde{\mathbb{E}}\left[D_T V_T\right] = \widetilde{\mathbb{E}}\left[e^{-\int_0^T r_t\,dt} V_T\right].$$

We recall/formalize the definition of $\widetilde{\mathbb{P}}$:

Definition 1.1 (Risk-neutral measure) A probability measure $\widetilde{\mathbb{P}}$ is called the *risk-neutral probability measure* if

1. $\widetilde{\mathbb{P}}$ is equivalent to $\mathbb{P}$, i.e., $\widetilde{\mathbb{P}} \sim \mathbb{P}$;
2. the discounted price of risky asset, $\{D_t S_t\}_{t\in[0,T]}$, is a $\widetilde{\mathbb{P}}$-martingale.

To construct such a $\widetilde{\mathbb{P}}$, we consider the Itô dynamics of $\{D_t S_t\}_{t\in[0,T]}$. By the product rule, we have

$$\begin{aligned} d(D_t S_t) &= D_t dS_t + S_t dD_t + d\langle S, D\rangle_t \\ &= D_t S_t \left((\mu_r - r_t)\,dt + \sigma_t\,dB_t\right) \\ &= D_t S_t d\widetilde{B}_t, \end{aligned}$$

where

$$\widetilde{B}_t = \int_0^t \theta_s\,ds,\ \ \theta_t := \frac{\mu_t - r_t}{\sigma_t}.$$

Therefore, if one can construct a probability measure $\widetilde{\mathbb{P}}$ such that $\widetilde{B}$ is a $\widetilde{\mathbb{P}}$-Brownian motion, $\{D_t S_t\}_{t\in[0,T]}$ would be a $\widetilde{\mathbb{P}}$-martingale. The construction is achieved by ***Girsanov's theorem***.

Let V_T be a $\mathcal{F}_T$-measurable random variable, which represents the payoff of a contingent claim written on S at time T. To derive the price of the contingent claim, we hope to construct a self-financing portfolio $\{X_t\}_{t\in[0,T]}$ using a hedging strategy with an adequate level of initial capital, X_0, such that, at $t \in [0,T]$

(i) hold Δ_t units of stock;

(ii) lend the amount $X_t - \Delta_t S_t$ at the risk-free rate r_t;

(iii) At $t = T$, $X_T = V_T$.

If such a strategy Δ_t exists, by the law of one price, the price of the contingent claim at any $t \in [0,T]$ is precisely X_t.

By the self-financing property, the dynamics of X is given by

$$\begin{aligned} dX_t &= \Delta_t dS_t + r_t(X_t - \Delta_t S_t)\,dt \\ &= \left[r_t X_t + \Delta_t S_t(\mu_t - r_t)\right]\,dt + \Delta_t S_t \sigma_t\,dB_t \\ &= r_t X_t\,dt + \Delta_t S_t \sigma_t\,d\widetilde{B}_t. \end{aligned}$$

The discounted portfolio value, $\{D_t X_t\}_{t\in[0,T]}$, thus follows

$$d(D_t X_t) = D_t\, dX_t + X_t\, dD_t + d\langle X, D\rangle_t = \Delta_t S_t \sigma_t\, d\widetilde{B}_t. \tag{2}$$

Two implications from the above derivations:

1. Under $\widetilde{\mathbb{P}}$, any portfolio strategy $\{\Delta_t\}_{t\in[0,T]}$ always earns a risk-free rate r_t;
2. the process $\{D_t X_t\}_{t\in[0,T]}$ is a $\widetilde{\mathbb{P}}$-martingale.

Owing to the second observation, we can express the portfolio value X_t (and thus the price of the contingent claim at t) as

$$X_t = \widetilde{\mathbb{E}}\left[\frac{D_T}{D_t} X_T | \mathcal{F}_t\right] = \widetilde{\mathbb{E}}\left[e^{-\int_t^T r_s\, ds} X_T | \mathcal{F}_t\right] = \widetilde{\mathbb{E}}\left[e^{-\int_t^T r_s\, ds} V_T | \mathcal{F}_t\right], \tag{3}$$

where the last equality follows from the assumption that, the strategy Δ is such that $X_T = V_T$.

This construction looks very natural, right? However, one major detail is indeed missing – is it always possible to find Δ in a way that $X_T = V_T$? The ***martingale representation theorem*** presents in the next subsection provide a positive answer to the construction.

1.2 Existence of Hedging Strategy

The risk-neutral pricing formula is based on the assumption that we can find a strategy $\{\Delta_t\}_{t\in[0,T]}$ such that the portfolio value $X_T = V_T$. The existence of such a portfolio strategy is a consequence of the martingale representation theorem (MRT):

Theorem 1.1 (Martingale representation) Let $\{B_t\}_{t\in[0,T]}$ be a Brownian motion on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t^B\}_{t\in[0,T]}, \mathbb{P})$, where $\{\mathcal{F}_t^B\}_{t\in[0,T]}$ is the filtration generated by $\{B_t\}_{t\in[0,T]}$, i.e., i.e.,

$$\mathcal{F}_t^B = \sigma\left(B_s : 0 \leq s \leq t\right).$$

Then, for any martingale $\{M_t\}_{t\in[0,T]}$ adapted to $\{\mathcal{F}_t^B\}_{t\in[0,T]}$, there exists an adapted process $\{\Gamma_t\}_{t\in[0,T]}$ such that

$$M_t = M_0 + \int_0^t \Gamma_s\, dB_s.$$

The MRT essentially says, if the filtration is generated by B, any $\mathbb{P}$-martingale can indeed be written as an Itô integral with an appropriate coefficient Γ and initial condition.

Based on Theorem 1.1, we have the following martingale representation theorem upon a change of measure:

Corollary 1.2 Let $\{B_t\}_{t\in[0,T]}$ be a Brownian motion on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t^B\}_{t\in[0,T]}, \mathbb{P})$. Let

$$\widetilde{B}_t := B_t + \int_0^t \theta_s\, ds,\;\; Z_t := \exp\left(-\int_0^t \theta_s\, dB_t - \frac{1}{2}\int_0^t \theta_s^2\, ds\right).$$

Then, for any $\{\mathcal{F}_t\}_{t\in[0,T]}$, $\widetilde{\mathbb{P}}$-martingale $\{\widetilde{M}_t\}_{t\in[0,T]}$, there exists an $\{\mathcal{F}_t^B\}_{t\in[0,T]}$-adapted process $\{\widetilde{\Gamma}_t\}_{t\in[0,T]}$ such that

$$\widetilde{M}_t = \widetilde{M}_0 + \int_0^t \widetilde{\Gamma}_s\, d\widetilde{B}_s.$$

Proof. We first claim that $M_t := Z_t\widetilde{M}_t$ is a $\mathbb{P}$-martingale. For any $0 \le s \le t \le T$, using Bayes' theorem of condition expectation (Lemma 3.4 in Chapter 7 of MATH5635),

$$\mathbb{E}[M_t|\mathcal{F}_s] = \mathbb{E}[Z_t\widetilde{M}_t|\mathcal{F}_s] = Z_s\mathbb{E}^{\widetilde{\mathbb{P}}}[\widetilde{M}_t|\mathcal{F}_s] = Z_s\widetilde{M}_s,$$

which verifies the claim. By Theorem 1.1, there exists an adapted process Γ such that

$$M_t = M_0 + \int_0^t \Gamma_s\, dB_s.$$

To proceed, we apply Itô's lemma on $\widetilde{M}_t := \frac{M_t}{Z_t}$. Recall

$$dZ_t = -Z_t\theta_t\, dB_t,$$

so that

$$d\left(\frac{1}{Z_t}\right) = -\frac{dZ_t}{Z_t^2} + \frac{1}{2}\left(\frac{2}{Z_t^3}\right) d\langle Z\rangle_t = \frac{\theta_t Z_t}{Z_t^2}\, dB_t + \frac{Z_t^2\theta_t^2}{Z_t^3}\, dt = \frac{1}{Z_t}\left(\theta_t^2\, dt + \theta_t\, dB_t\right).$$

Hence, by the product rule,

$$\begin{aligned}
d\widetilde{M}_t &= d\left(\frac{M_t}{Z_t}\right)\\
&= \frac{dM_t}{Z_t} + M_t d\left(\frac{1}{Z_t}\right) + d\left\langle M, \frac{1}{Z_t}\right\rangle\\
&= \frac{\Gamma_t}{Z_t}\, dB_t + \frac{M_t}{Z_t}\left(\theta_t^2\, dt + \theta_t\, dB_t\right) + \frac{\theta_t\Gamma_t}{Z_t}\, dt\\
&= \frac{\theta_t}{Z_t}\left(M_t\theta_t + \Gamma_t\right)\, dt + \frac{1}{Z_t}\left(\Gamma_t + \theta_t M_t\right)\, dB_t\\
&= \frac{\theta_t}{Z_t}\left(M_t\theta_t + \Gamma_t\right)\, dt + \frac{1}{Z_t}\left(\Gamma_t + \theta_t M_t\right)\left[d\widetilde{B}_t - \theta_t\, dt\right]
\end{aligned}$$

$$= \frac{\Gamma_t + \theta_t M_t}{Z_t}\, d\widetilde{B}_t$$
$$= \left(\frac{\Gamma_t}{Z_t} + \widetilde{M}_t\right) d\widetilde{B}_t.$$

Therefore, we arrive at the representation with

$$\widetilde{\Gamma}_t := \frac{\Gamma_t}{Z_t} + \widetilde{M}_t.$$

□

The existence of $\{\Delta_t\}_{t\in[0,T]}$ can now be proven using Corollary 1.2. We define a process $\{\widetilde{V}_t\}_{t\in[0,T]}$ by

$$\widetilde{V}_t := \widetilde{\mathbb{E}}\left[\frac{D_T}{D_t} V_T | \mathcal{F}_t\right]. \tag{4}$$

This is exactly the equation satisfied by X in (3). However, unlike the derivation of (3), the process $\{\widetilde{V}_t\}_{t\in[0,T]}$ is defined independently of any trading strategy, which is always possible. By construction, $\{D_t\widetilde{V}_t\}_{t\in[0,T]}$ is a $\widetilde{\mathbb{P}}$-martingale. Hence, by the MRT (Corollary 1.2), there exists an adapted process $\widetilde{\Gamma}$ such that

$$D_t\widetilde{V}_t = \widetilde{V}_0 + \int_0^t \widetilde{\Gamma}_s\, d\widetilde{B}_s. \tag{5}$$

Now, define the initial capital $X_0 := \widetilde{V}_0$, and the hedging strategy by

$$\Delta_t := \frac{\widetilde{\Gamma}_t}{S_t\sigma_t}. \tag{6}$$

Substituting this and the choice of initial capital $X_0 = \widetilde{V}_0$ into the dynamics (2), we have

$$D_tX_t = X_0 + \int_0^t \Delta_s S_s \sigma_s\, d\widetilde{B}_s = \widetilde{V}_0 + \int_0^t \widetilde{\Gamma}_s\, d\widetilde{B}_s.$$

which is precisely (5). Hence, $X_t = \widetilde{V}_t$ for all $t \in [0,T]$. In particular, $X_T = \widetilde{V}_T = V_T$. Therefore, the desired hedging strategy is indeed given by (6).

The MRT only tells us the a hedging portfolio exists and is related to the process $\widetilde{\Gamma}$. However, it does not give the explicit form of the process $\widetilde{\Gamma}$ in general. Recall that in the particular case when we were pricing European options under the Black-Scholes model, i.e., $\mu_\cdot \equiv \mu$ and $\sigma_\cdot \equiv \sigma > 0$ and $V_T = V(T, S_T)$, the hedging portfolio is given by

$$\Delta_t = V_S(t, S_t),$$

where $V(t, s)$ represents the price of the European option at time t when $S_t = s$, which is the solution of the ***Black-Scholes partial differential equation*** (PDE). Hence, the corresponding process $\widetilde{\Gamma}$ is given by

$$\widetilde{\Gamma}_t = S_t \sigma_t \Delta_t = S_t \sigma_t V_S(t, S_t).$$

In the next chapter, we will further relate the hedging strategy with solutions of PDEs.

2 Multidimensional Pricing Model

In this section, we consider a market with m risky assets driven by a d-dimensional Brownian motion in the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$,

$$\boldsymbol{B}_t = (B_t^1, \dots, B_t^d),\ t \in [0, T],$$

such that each B^i, $i = 1, \dots, d$, is a standard 1-dimensional Brownian motion, each B^i, B^j are independent for $i \neq j$.

We write $\boldsymbol{S}_t = (S_t^1, \dots, S_t^m)$, where S_t^i represents the price of the i-th risky asset at time t, which is governed by the following dynamics:

$$dS_t^i = \mu_t^i S_t^i \, dt + S_t^i \sum_{j=1}^{d} \sigma_t^{ij} \, dB_t^j. \tag{7}$$

Here, $(\mu^i)_{i=1}^m$, $(\sigma^{ij})_{i=1,\dots,m,j=1,\dots,d}$ are $\{\mathcal{F}_t\}$-adapted processes representing the rate of return vector and the volatility matrix, respectively.

Our objective is to construct risk-neutral measures and replicating portfolios for contingent claims in a multi-dimensional pricing model. Unlike the one-dimensional case, this requires additional conditions on the drift and volatility coefficients. We begin by extending the definition of a risk-neutral probability measure to the multi-dimensional setting.

Definition 2.1 (Risk-neutral measure for multidimensional models) A probability measure $\widetilde{\mathbb{P}}$ is called a *risk-neutral probability measure* if

1. $\widetilde{\mathbb{P}}$ is equivalent to $\mathbb{P}$, i.e., $\widetilde{\mathbb{P}} \sim \mathbb{P}$;
2. for $i = 1, \dots, m$, the discounted price of risky asset, $\{D_t S_t^i\}_{t\in[0,T]}$, is a $\widetilde{\mathbb{P}}$-martingale.

In the sequel, we shall need the following generalization of Girsanov's theorem and MRT for multidimensional models, which we state without proof. Nevertheless, we remark that the multidimensional Girsanov's theorem can be proven using Lévy's characterization.

Theorem 2.1 (Multidimensional Girsanov's) Let $\{\boldsymbol{B}_t\}_{t\in[0,T]}$ be a d-dimensional Brownian motion in the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$, and $\{\boldsymbol{\theta}_t\}_{t\in[0,T]}$ be a d-dimensional adapted process of the form $\boldsymbol{\theta}_t = (\theta_t^1, \ldots, \theta_t^d)$. Define the processes

$$\begin{aligned} Z_t &:= \exp\left(-\int_0^t \boldsymbol{\theta}_s \cdot d\boldsymbol{B}_s - \frac{1}{2}\int_0^t |\boldsymbol{\theta}_s|^2\, ds\right) \\ &= \exp\left(-\sum_{j=1}^d \int_0^t \theta_s^j\, dB_s^j - \frac{1}{2}\int_0^t \sum_{j=1}^d |\theta_s^j|^2\, ds\right), \\ \widetilde{\boldsymbol{B}}_t &:= \boldsymbol{B}_t + \int_0^t \boldsymbol{\Theta}_{\boldsymbol{s}}\, ds = \left(B_t^1 + \int_0^t \theta_s^1\, ds, \cdots, B_t^d + \int_0^t \theta_s^d\, ds\right). \end{aligned}$$

Suppose that

$$\mathbb{E}\left[\int_0^T |\boldsymbol{\theta}_t|^2 Z_t^2\, dt\right] < \infty.$$

Then, $\mathbb{E}[Z_T] = 1$, and under the probability measure $\widetilde{\mathbb{P}}$, where

$$\widetilde{\mathbb{P}}(A) := \mathbb{E}[Z_T \mathbb{1}_A],\ \ A \in \mathcal{F}_T = \mathcal{F},$$

the process $\{\widetilde{\boldsymbol{B}}_t\}_{t\in[0,T]}$ is a standard Brownian motion.

Theorem 2.2 (Multidimensional martingale representation) Let $\{\boldsymbol{B}_t\}_{t\in[0,T]}$ be a d-dimensional Brownian motion on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t^{\boldsymbol{B}}\}_{t\in[0,T]}, \mathbb{P})$, where $\{\mathcal{F}_t^{\boldsymbol{B}}\}_{t\in[0,T]}$ is the filtration generated by $\{\boldsymbol{B}_t\}_{t\in[0,T]}$, i.e., i.e.,

$$\mathcal{F}_t^{\boldsymbol{B}} = \sigma\left(\boldsymbol{B}_s : 0 \le s \le t\right).$$

Then, for any martingale $\{M_t\}_{t\in[0,T]}$ adapted to $\{\mathcal{F}_t^{\boldsymbol{B}}\}_{t\in[0,T]}$, there exists a d-dimensional adapted process $\{\boldsymbol{\Gamma}_t\}_{t\in[0,T]}$ with $\boldsymbol{\Gamma}_t = (\Gamma_t^1, \ldots, \Gamma_t^d)$ such that

$$M_t = M_0 + \int_0^t \boldsymbol{\Gamma}_s \cdot d\boldsymbol{B}_s.$$

In addition, using the notations and assumptions in Theorem 2.1, for any $\widetilde{\mathbb{P}}$-martingale $\{\widetilde{M}_t\}_{t\in[0,T]}$ adapted to $\{\mathcal{F}_t^{\boldsymbol{B}}\}_{t\in[0,T]}$, there exists a d-dimensional adapted process $\{\widetilde{\boldsymbol{\Gamma}}_t\}_{t\in[0,T]}$ with $\widetilde{\boldsymbol{\Gamma}}_t = (\widetilde{\Gamma}_t^1, \ldots, \widetilde{\Gamma}_t^d)$ such that

$$\widetilde{M}_t = \widetilde{M}_0 + \int_0^t \widetilde{\boldsymbol{\Gamma}}_s \cdot d\widetilde{\boldsymbol{B}}_s.$$

3 First Fundamental Theorem of Asset Pricing

This section discusses the existence of risk-neutral probability measure under the multidimensional market model and its relationship with the no-arbitrage property of the market. The latter is framed as the ***first fundamental theorem of asset pricing***.

Recall in the 1-dimensional case, to make $\{D_t S_t\}_{t\in[0,T]}$ a $\widetilde{\mathbb{P}}$-martingale, we "absorb" the drift term of $D_t S_t$ into the stochastic integral:

$$d(D_t S_t) = D_t S_t(\mu_t - r_t)\,dt + D_t S_T \sigma_t\,dB_t = D_t S_t \sigma_t\left(\theta_t\,dt + dB_t\right),$$

and we define $\widetilde{B}_t$ such that $d\widetilde{B}_t = \theta_t\,dt + dB_t$. In the multidimensional case, we follow a similar path, except that the market price of risk would be a d-dimensional vector, which shall be solved by a system of linear equations.

For $i = 1, \ldots, m$, by the product rule, we have

$$\begin{aligned} d(D_t S_t^i) &= S_t^i\,dD_t + D_t\,dS_t^i \\ &= -r_t D_t S_t^i\,dt + D_t S_t^i\left(\mu_t^i\,dt + \sum_{j=1}^d \sigma_t^{ij}\,dB_t^j\right) \\ &= D_t S_t^i\left[(\mu_t^i - r_t)\,dt + \sum_{j=1}^d \sigma_t^{ij}\,dB_t^j\right]. \end{aligned} \tag{8}$$

Suppose that there exists a d-dimensional process $\boldsymbol{\theta} = (\theta^1, \ldots, \theta^d)$ such that,

$$\mu_t^i - r_t = \sum_{j=1}^d \sigma_t^{ij}\theta_t^j,\ i = 1, \ldots, m,\ t \in [0, T]. \tag{9}$$

Then, we can rewrite (8) as

$$\begin{aligned} d(D_t S_t^i) &= D_t S_t^i\left[\sum_{j=1}^d \sigma_t^{ij}\theta_t^j\,dt + \sum_{j=1}^d \sigma_t^{ij}\,dB_t^j\right] \\ &= D_t S_t^i \sum_{j=1}^d \sigma_t^{ij}\left[\theta_t^j\,dt + dB_t^j\right] \\ &= D_t S_t^i \sum_{j=1}^d \sigma_t^{ij}\,d\widetilde{B}_t^j, \end{aligned} \tag{10}$$

where $\widetilde{B}_t^j := B_t^j + \int_0^t \theta_s^j\,ds$. By Girsanov's theorem (Theorem 2.1), we can define $\widetilde{\mathbb{P}}$ such that $\widetilde{\boldsymbol{B}}_t = (\widetilde{B}_t^1, \ldots, \widetilde{B}_t^d)$ is a d-dimensional $\widetilde{\mathbb{P}}$-Brownian motion, making each $D_t S_t^i$, $i = 1, \ldots, m$ a $\widetilde{\mathbb{P}}$-martingale. Consequently, $\widetilde{\mathbb{P}}$ is a risk-neutral probability measure.

Therefore, the existence of a risk-neutral probability measure amounts to the existence of solutions for the linear system (9) for $t \in [0, T]$, which we shall refer to as the ***market price of risk equations***. One can also express the system in matrix form: define

$$\mathbf{\Sigma}_t := \begin{pmatrix} \sigma_t^{11} & \cdots & \sigma_t^{1d} \\ \vdots & \ddots & \vdots \\ \sigma_t^{m1} & \cdots & \sigma_t^{md} \end{pmatrix} \in \mathbb{R}^{m \times d}, \quad \boldsymbol{\alpha}_t = \begin{pmatrix} \mu_t^1 - r_t \\ \vdots \\ \mu_t^m - r_t \end{pmatrix} \in \mathbb{R}^m. \tag{11}$$

Then, the system is equivalent to solving $\boldsymbol{\theta}_t$ such that

$$\mathbf{\Sigma}_t \boldsymbol{\theta}_t = \boldsymbol{\alpha}_t, \ \ t \in [0, T]. \tag{12}$$

Note that for each $t \in [0, T]$, the system has m equations with d unknowns.

Theorem 3.1 A risk-neutral probability measure exists if the market price of risk equations (9) (or equivalently, (12)) admits a solution for all $t \in [0, T]$. Equivalently, such a measure exists if, for any $t \in [0, T]$,

$$\boldsymbol{\alpha}_t \in \operatorname{Col}(\mathbf{\Sigma}_t),$$

where $\operatorname{Col}(\boldsymbol{A})$ denotes the column space of a matrix $\boldsymbol{A}$.

A sufficient condition for existence of a risk-neutral probability measure is $\operatorname{rank}(\mathbf{\Sigma}_t) = d$ for all $t \in [0, T]$. Theorem 3.1 only discuss conditions for existence of risk-neutral measures, but not uniqueness. Indeed, there may exist multiple risk-neutral probability measures, and the implications shall be discussed in the next section.

Consider the portfolio value process $\{X_t\}_{t \in [0,T]}$ with the self-financing portfolio strategy $\mathbf{\Delta}_t = (\Delta_t^1, \ldots, \Delta_t^m)$ with an initial capital X_0, where:

1. Δ_t^i denotes the number of shares held in the i-th risky asset with price S_t^i. The total market value invested in the risky assets $\boldsymbol{S}$ is thus $\sum_{i=1}^m \Delta_t^i S_t^i$;
2. the remainder of the portfolio value earns a risk-free rate r_t.

Hence, the dynamics of X_t is given by

$$\begin{aligned} dX_t &= \sum_{i=1}^m \Delta_t^i \, dS_t^i + r_t \left(X_t - \sum_{i=1}^m \Delta_t^i S_t^i \right) dt \\ &= r_t X_t \, dt + \sum_{i=1}^m \Delta_t^i \left(dS_t^i - r_t S_t^i \right) \, dt \\ &= r_t X_t \, dt + \sum_{i=1}^m \frac{\Delta_t^i}{D_t} \, d(D_t S_t^i). \end{aligned} \tag{13}$$

Hence, the discounted portfolio value, D_tX_t, has the following dynamics:

$$d(D_tX_t) = X_t\,dD_t + D_t\,dX_t = \sum_{i=1}^{m} \Delta_t^i\,d(D_tS_t^i). \tag{14}$$

If a risk-neutral probability measure $\widetilde{\mathbb{P}}$ exists such that $D_tS_t^i$, $i = 1, \ldots, m$, are $\widetilde{\mathbb{P}}$-martingales. Then, from (2) and (14), we have the following observations, similar to the 1-dimensional case:

1. the portfolio X_t earns the risk-free rate r_t under $\widetilde{\mathbb{P}}$;
2. the discounted portfolio value D_tX_t is a $\widetilde{\mathbb{P}}$-martingale.

The existence of a risk-neutral probability measure asserts that the market is free from ***arbitrage*** opportunity, which is framed in the ***first fundamental theorem of asset pricing***. We first define the meaning of arbitrage:

Definition 3.1 An ***arbitrage*** is a portfolio X_t such that $X_0 = 0$, and there exists $T > 0$ such that

$$\mathbb{P}(X_T \geq 0) = 1,\ \mathbb{P}(X_T > 0) > 0.$$

In other words, an arbitrage is a self-financing portfolio with zero initial capital that never incurs a loss and yields a strictly positive payoff with positive probability.

The following theorem is the main result of this section, which states that a market is arbitrage-free if $\widetilde{\mathbb{P}}$ exists.

Theorem 3.2 (First fundamental theorem of asset pricing) If there exists a risk-neutral probability measure for the market model, then the market admits no arbitrage.

Proof. Let X_t be a portfolio with zero initial capital, i.e., $X_0 = 0$. If $\widetilde{\mathbb{P}}$ exists, by the above discussion, we know that D_tX_t is a $\widetilde{\mathbb{P}}$-martingale.

Assume that contrary that X_t is an arbitrage, i.e., there exists $T > 0$ such that $\mathbb{P}(X_T \geq 0) = 1$, $\mathbb{P}(X_T > 0) > 0$. By the equivalence of $\mathbb{P}$ and $\widetilde{\mathbb{P}}$, we also have $\widetilde{\mathbb{P}}(X_T \geq 0) = 1$, $\widetilde{\mathbb{P}}(X_T > 0) > 0$. Since $D_T > 0$ a.s., we have $D_TX_T \geq 0$ $\mathbb{P}$-a.s., and $\widetilde{P}(D_TX_T > 0) > 0$. This would imply

$$\widetilde{\mathbb{E}}[D_TX_T] > 0.$$

However, using the martingale property of D_tX_t under $\widetilde{\mathbb{P}}$ and the fact that $X_0 = 0$, we have

$$\widetilde{\mathbb{E}}[D_TX_T] = D_0X_0 = 0,$$

which is absurd. Hence, no arbitrage exists. □

The following example illustrates that arbitrage could exist if (9) is not solvable.

Example 3.1 Suppose that there are 2 stocks with 1 driving Brownian motion, i.e., $m = 2$ and $d = 1$. In addition, we assume that all model parameters, μ, r, σ, are constants. As such, the system (9) is reduced to a 2×1 system:

$$\begin{cases} \dfrac{\mu^1 - r}{\sigma^1} = \theta, \\ \dfrac{\mu^2 - r}{\sigma^2} = \theta. \end{cases}$$

The system is solvable iff

$$\frac{\mu^1 - r}{\sigma^1} = \frac{\mu^2 - r}{\sigma^2}.$$

Suppose that the above condition does not hold, and without loss of generality, suppose that $\frac{\mu^1-r}{\sigma^1} < \frac{\mu^2-r}{\sigma^2}$. Then, consider a portfolio with $\Delta_t^1 = -\frac{1}{S_t^1\sigma^1}$ units of S^1 (short), and $\Delta_t^2 = \frac{1}{S_t^2\sigma^2}$ units of S^2 (long). The amount of initial capital required would be

$$\Delta_0^1 S_0^1 + \Delta_0^2 S_0^2 = \frac{1}{\sigma^2} - \frac{1}{\sigma^1}.$$

We borrow (resp. lend) at the risk-free rate if $\frac{1}{\sigma^2} - \frac{1}{\sigma^1}$ is positive (resp. negative). In that case, the initial capital is $X_0 = 0$.
For $t > 0$, the dynamics of X is given by

$$\begin{aligned} dX_t &= \sum_{i=1}^{2} \Delta_t^i dS_t^i + r(X_t - \Delta_t^1 S_t^1 - \Delta_t^2 S_t^2)\, dt \\ &= -\frac{1}{\sigma^1}\left(\mu^1\, dt + \sigma^1\, dB_t\right) + \frac{1}{\sigma^2}\left(\mu^2\, dt + \sigma^2\, dB_t\right) + r\left(X_t + \frac{1}{\sigma^1} - \frac{1}{\sigma^2}\right) dt \\ &= rX_t\, dt + \left[\frac{\mu^2 - r}{\sigma^2} - \frac{\mu^1 - r}{\sigma^1}\right] dt. \end{aligned}$$

Hence, the discounted portfolio value is

$$d(D_t X_t) = D_t \left[\frac{\mu^2 - r}{\sigma^2} - \frac{\mu^1 - r}{\sigma^1}\right] dt.$$

By the given assumption, $D_t X_t$ is non-random with a positive drift, whence it admits a positive and non-random portfolio value with zero initial capital, i.e., arbitrage.

The existence of RNMs is characterized by the existence of solutions to the system (12). Below we examine the relationship between the existence and the values of m and d.

- Case 1: $m > d$

- More assets than sources of randomness, the system (12) is overdetermined;
- Risk-neutral measure exists only if the asset drifts are mutually consistent (see Example 3.1 for inconsistent asset drifts).

- Case 2: $m < d$
  - Fewer risky assets than sources of randomness, the system (12) is underdetermined;
  - The system either has no solution (inconsistent drifts; no RNM), or infinitely many solutions (consistent drifts; multiple RNMs)
- Case 3: $m = d$
  - If $\boldsymbol{\Sigma}_t$ is invertible, then RNM exists and is unique;
  - If $\boldsymbol{\Sigma}_t$ is not invertible, then either RNM does not exists, or there exists infinitely many RNMs.

Summary: The existence of RNMs is "easier" when $m \leq d$, since *more risky assets may lead to inconsistent drift conditions.*

4 Second Fundamental Theorem of Asset Pricing

In this section, we discuss the uniqueness of risk-neutral probability measure and its implication – market completeness. This result is depicted in the ***second fundamental theorem of asset pricing***.

Definition 4.1 A market is said to be ***complete*** if, for any contingent claim with payoff $V_T \in L^2(\mathcal{F}_T)$, there exists a self-financing portfolio strategy $\boldsymbol{\Delta}_t = (\Delta_t^1 \ldots, \Delta_t^m)$ and an initial capital X_0 such that

$$X_T = V_T, \quad \mathbb{P}\text{-a.s.}$$

Recall that if a risk-neutral probability measure $\widetilde{\mathbb{P}}$ exists, the risk-neutral price of the contingent claim at time t is given by

$$V_t := \widetilde{\mathbb{E}}\left[\frac{D_T}{D_t} V_t \big| \mathcal{F}_t\right]. \tag{15}$$

By the law of one-price (or the first fundamental theorem of asset pricing), in that case, a market is complete iff there exists a self-financing portfolio strategy $\boldsymbol{\Delta}_t$ and an initial capital X_0 such that for any $t \in [0, T]$,

$$X_t = V_t, \ \mathbb{P}\text{-a.s}$$

In the sequel, we assume that the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ is generated by the d-dimensional Brownian motion $\boldsymbol{B}$, i.e., $\mathcal{F}_t = \mathcal{F}_t^{\boldsymbol{B}}$. We also assume that a risk-neutral probability $\widetilde{\mathbb{P}}$ exists.

Since $\{D_t V_t\}_{t\in[0,T]}$ defined in (15) is a $\widetilde{\mathbb{P}}$-martingale, by the multidimensional MRT (Theorem 2.2), there exists a d-dimensional process $\widetilde{\boldsymbol{\Gamma}}_t = (\widetilde{\Gamma}_t^1, \dots, \widetilde{\Gamma}_t^d)$ such that

$$D_t V_t = V_0 + \sum_{j=1}^{d} \int_0^t \widetilde{\Gamma}_s^j \, d\widetilde{B}_s^j. \tag{16}$$

On the other hand, consider a portfolio with initial capital X_0 and self-financing portfolio strategy $\boldsymbol{\Delta}_t$, using (14) and (10), its dynamics is given by

$$\begin{aligned} D_t X_t &= D_0 X_0 + \sum_{i=1}^{m} \int_0^t \Delta_s^i \, d(D_s S_s^i) \\ &= D_0 X_0 + \sum_{j=1}^{d} \int_0^t \sum_{i=1}^{m} \Delta_s^i D_s S_s^i \sigma_s^{ij} \, d\widetilde{B}_s^j, \end{aligned} \tag{17}$$

By equating (16) and (17), the desired portfolio strategy can be obtain by solving the following system of equation:

$$\sum_{i=1}^{m} \Delta_t^i S_t^i \sigma_t^{ij} = \frac{\widetilde{\Gamma}_t^j}{D_t}, \;\; j = 1, \dots, d, \;\; t \in [0, T]. \tag{18}$$

Note that for each $t \in [0, T]$, (18) is a linear system with d equations and m unknowns. By letting

$$\boldsymbol{\beta}_t := \begin{pmatrix} \frac{\widetilde{\Gamma}_t^1}{D_t} \\ \vdots \\ \frac{\widetilde{\Gamma}_t^d}{D_t} \end{pmatrix} \in \mathbb{R}^d, \;\; \boldsymbol{y}_t := \begin{pmatrix} \Delta_t^1 S_t^1 \\ \vdots \\ \Delta_t^m S_t^m \end{pmatrix} \in \mathbb{R}^m,$$

the system (18) can also be written as

$$\boldsymbol{\Sigma}_t^\top \boldsymbol{y}_t = \boldsymbol{\beta}_t, \;\; t \in [0, T], \tag{19}$$

where $\boldsymbol{\Sigma}_t \in \mathbb{R}^{m\times d}$ was defined in (11). Hence, the contingent claim can be hedged in (19) admits a solution.

The following is the main result of this section, which relates market completeness with uniqueness of risk-neutral measures.

Theorem 4.1 (Second fundamental theorem of asset pricing) Suppose the risk-neutral measures exist in a market model. Then, the market is complete if and only if the risk-neutral measure is unique.

Proof. Suppose that the market is complete. We shall show the the risk-neutral probability measure is unique.

Let $\widetilde{\mathbb{P}}^1$ and $\widetilde{\mathbb{P}}^2$ be two risk-neutral measures, and denote the associated expected values by $\widetilde{\mathbb{E}}^1$ and $\widetilde{\mathbb{E}}^2$, respectively. Let $A \in \mathcal{F}_T = \mathcal{F}_T^{\boldsymbol{B}}$, and consider a contingent claim with payoff $V_T = \frac{\mathbb{1}_A}{D_T}$. Since the market is complete, under each $\widetilde{\mathbb{P}}^i$, $i = 1, 2$, there exists a portfolio with initial capital X_0^i and a self-financing strategy such that $X_T^i = V_T$, $i = 1, 2$, and that $D_t X_t^i$ is a $\widetilde{\mathbb{P}}^i$-martingale. Hence,

$$
\begin{aligned}
X_0^1 &= \widetilde{\mathbb{E}}^1[D_T X_T^1] = \widetilde{\mathbb{E}}^1[D_T V_T] = \widetilde{\mathbb{P}}^1(A), \\
X_0^2 &= \widetilde{\mathbb{E}}^1[D_T X_T^2] = \widetilde{\mathbb{E}}^1[D_T V_T] = \widetilde{\mathbb{P}}^2(A).
\end{aligned}
$$

Note that by the first fundamental theorem of asset pricing, no arbitrage could exist, and thus $X_0^1 = X_0^2$. Hence, $\widetilde{\mathbb{P}}^1(A) = \widetilde{\mathbb{P}}^2(A)$. Since $A \in \mathcal{F}_T$ is arbitrary, we conclude that $\widetilde{\mathbb{P}}^1 \equiv \widetilde{\mathbb{P}}^2$.

Next, we suppose that the risk-neutral probability measure $\widetilde{\mathbb{P}}$ is unique. This is equivalent to requiring that the system (12) admits a unique solution for each $t \in [0, T]$. Hence, for any $t \in [0, T]$, $\text{null}(\boldsymbol{\Sigma}_t) = \dim(\text{Ker}(\boldsymbol{\Sigma}_t)) = 0$, where

$$
\text{Ker}(\boldsymbol{A}) := \{\boldsymbol{x} \in \mathbb{R}^d : \boldsymbol{A}\boldsymbol{x} = 0\}.
$$

By the rank-nullity theorem, we have $\text{rank}(\boldsymbol{\Sigma}_t) = d - 0 = d$, for any $t \in [0, T]$, which also implies $\text{rank}(\boldsymbol{\Sigma}_t^\top) = d$. Consequently, the system (19) is solvable for any given $\boldsymbol{\beta}_t \in \mathbb{R}^d$. In other words, one can always find a hedging strategy that replicates the payoff of any contingent claim. Therefore, the market is complete.

□

We examine the relationship between the uniqueness of a RNM and the values of m and d, which is characterized by the system (19).

- Case 1: $m > d$
  - More risky assets than sources of randomness; the system (19) is under-determined;
  - If a risk-neutral measure exists, then it is unique if and only if $\text{rank}(\boldsymbol{\Sigma}_t) = d$ for all $t \in [0, T]$.
- Case 2: $m < d$
  - Fewer risky assets than sources of randomness, the system (19) is over-determined;
  - Since $\text{rank}(\boldsymbol{\Sigma}_t) \leq m < d$, there exists contingent claims such that (19) is not solvable;
  - The market is not complete.
- Case 3: $m = d$

– If $\mathbf{\Sigma}_t$ is invertible, then RNM exists and is unique;

– If $\mathbf{\Sigma}_t$ is not invertible, then either RNM does not exists, or there exists infinitely many RNMs.

Summary: The uniqueness of RNMs is "easier" when $m \geq d$, since *more risky assets can be used to hedge the sources of randomness.*

Further Readings

1. Novikov's condition and its relation with Girsanov's theorem;
2. Risk-neutral pricing under incomplete market, i.e., multiple risk-neutral measures;
3. Pricing of forwards and futures.